1. *Let $f : [0,1] \to \mathbb{R}$ be the function $f(x) = x \ \forall \ x \in [0,1]$. From first principles, that is, through computing upper and lower sums, show that*
$$\int_0^1 f = \frac{1}{2}.$$

Solution: Let P_N be the partition $\{0, \frac{1}{N}, \frac{2}{N}, ..., 1\}$, i.e. each point is evenly spaced with the distance $\frac{1}{N}$. The upper and lower sums for such a partition are

$$U(P_N, f) = \sum_{i=1}^{N} \sup_{[\frac{i-1}{N}, \frac{i}{N}]} x \ [\frac{i}{N} - \frac{i-1}{N}] = \sum_{i=1}^{N} \frac{i}{N}\frac{1}{N} = \frac{1}{N^2}[\frac{N(N+1)}{2}] = \frac{1}{2} + \frac{1}{2N}.$$

$$L(P_N, f) = \sum_{i=1}^{N} \inf_{[\frac{i-1}{N}, \frac{i}{N}]} x \ [\frac{i}{N} - \frac{i-1}{N}] = \sum_{i=1}^{N} \frac{i-1}{N}\frac{1}{N} = \frac{1}{N^2}[\frac{N(N-1)}{2}] = \frac{1}{2} - \frac{1}{2N}.$$

Let $N \to \infty$, both $U(P_N, f)$ and $L(P_N, f)$ go to $\frac{1}{2}$, from above and below respectively. Since upper sums are upper bounds for lower sums, every lower sum is bounded above by $\frac{1}{2}$. Since there are lower sums that are arbitrarily close to $\frac{1}{2}$ (take N large enough), it follows that $\frac{1}{2}$ is the least upper bound for the lower sums. Similarly we conclude that $\frac{1}{2}$ is the greatest lower bound for upper sums. This shows that $f(x)$ is Riemann integrable on $[0,1]$ with integral $\frac{1}{2}$.

□

2. *Suppose $g : [0,1] \to [0, \infty)$ is a continuous function and $\int_0^1 g = 0$ for all $t \in [0,1]$.*

Solution: Suppose $g \neq 0$ on $[0,1]$. Since g is continuous, there is a non empty interval $(c,d) \subseteq [0,1]$ and $\alpha > 0$ satisfying $g(x) \geq \frac{\alpha}{2}$ for $x \in (c,d)$, then

$$\int_0^1 g \geq \int_c^d g \geq \frac{\alpha}{2}(d-c) > 0.$$

This is a contradiction to $\int_0^1 g = 0$.

□

3. *Let Ω be a non-empty finite set and let $\mathcal{F} = \{A : A \subseteq \Omega\}$ be the power set of Ω. For $A, B \in \mathcal{F}$, define $d(A,B)$ as the number of elements in $A\Delta B := (A \cap B^c) \cup (B \cap A^c)$. Show that d is a metric on $\mathcal{F}$.*

Solution: (i). Since $A\Delta A = (A \cap A^c) \cup (A \cap A^c) = \emptyset$, therefore $d(A,A) = 0$. Suppose $d(A,B) = 0$, then $(A \cap B^c) \cup (B \cap A^c) = \emptyset$, we have both $(A \cap B^c) = \emptyset$ and $(B \cap A^c) = \emptyset$. Therefore $A = B$.

(ii). $A\Delta B = (A \cap B^c) \cup (B \cap A^c) = (B \cap A^c) \cup (A \cap B^c) = B\Delta A$ implies $d(A,B) = d(B,A)$.

(iii). $d(A,C) \leq d(A,B) + d(B,C)$, Using Venn diagrams one can easily verify the triangular inequality.

□

4. *Let (X, d) be a compact metric space. Suppose $f : X \to \mathbb{R}$ is a continuous function. Show that $\{f(x) : x \in \mathbb{X}\}$ is compact.*

Solution: Let x_n be any sequence in X. Since X is compact then x_n has a convergent subsequence x_q. Now take the sequence $f(x_n)$ and notice that it has a convergent subsequence $f(x_q)$ [using definition of continuity]. Since for any point $y \in f(X)$, there exists $x \in X$ such that $f(x) = y$. Every sequence in $f(X)$ can be written as $f(x_n)$ for some sequence $x_n \in X$. Therefore $f(X)$ is compact.

$\square$

5. *Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$h(x, y) = \begin{cases} \frac{3x^4}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Determine as to whether h is continuous at the origin or not. Compute partial derivatives $D_1h(0, 0)$ and $D_2h(0, 0)$ if they exist.

Solution: To prove that h is continuous at the origin. Let $\epsilon > 0$ be given, choose $\delta > 0$ such that $\delta^2 = \frac{\epsilon}{3}$. Whenever $0 < (x, y) \in \mathbb{R}^2$ with $|(x, y)| := \max\{|x|, |y|\} < \epsilon$, we have

$$|h(x, y) - h(0, 0)| = |h(x, y)| = \frac{3x^4}{x^2 + y^2} \leq \frac{3x^4}{x^2} = 3x^2 < 3\delta^2 < \epsilon.$$

Thus h is continuous at $(0, 0)$.

$$D_1h(0, 0) = \lim_{t\to 0} \frac{h(0 + t, 0) - h(0, 0)}{t} = \lim_{t\to 0} \frac{h(t, 0) - h(0, 0)}{t} = \lim_{t\to 0} 3t = 0$$

$$D_2h(0, 0) = \lim_{t\to 0} \frac{h(0, 0 + t) - h(0, 0)}{t} = \lim_{t\to 0} \frac{h(0, t) - h(0, 0)}{t} = \lim_{t\to 0} \frac{0 - 0}{t} = 0$$

$\square$

6. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable (has total derivative) at $a \in \mathbb{R}^n$. Show that f is continuous at $a \in \mathbb{R}^n$.*

Solution: Suppose f is differentiable at a, then there exists $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$, such that

$$|f(a + h) - f(a) - \alpha h| = ||h||\epsilon(h) \quad \text{and} \quad \epsilon(h) \to 0 \quad \text{as} \quad h \to 0.$$

Hence

$$|f(a + h) - f(h)| \leq ||h||(\sum_{i=1}^{n} |\alpha_i|) + ||h||\epsilon(h).$$

and $\epsilon(h) \to 0$ as $h \to 0$, therefore $f(a + h) - f(a)$ as $h \to 0$. This proves that f is continuous at a.

$\square$

7. *Use the method of Lagrange multipliers to find the point nearest to the origin in the plane* $2x+3y-z=5$ *in* $\mathbb{R}^3$.

Solution: The distance of an arbitrary point (x, y, z) from the origin is $d = \sqrt{x^2+y^2+z^2}$. It is geometrically clear that there is an absolute minimum of this function for (x, y, z) lying on the plane. To find it, we instead minimize the function

$$d^2 = f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint $g(x, y, z) = 0$ where $g(x, y, z) = 2x + 3y - z - 5$. The gradients of these two functions are $\nabla f = (2x, 2y, 2z)$, $\nabla g = (2, 3, -1)$. Since $\nabla g \neq 0$ ever, the absolute minimum of the distance function we are looking for will occur at a point where

$$\nabla f = \lambda g, \quad g = 0$$

Getting rid of the 2's in ∇f (which all get absorbed into the dummy constant λ) and settings components of the gradient equation, we obtain the system of equations,

$$x = 2\lambda, y = 3\lambda, z = -\lambda, 2x + 3y - z = 5.$$

Solving the first three equations gives $y = \frac{3}{2}x$, $z = \frac{-1}{2}x$. Plugging these into the equation of the plane gives $2x + \frac{9}{2}x + \frac{1}{2}x = 5$, and so the point we are looking for is $x = \frac{5}{7}, y = \frac{15}{14}, z = \frac{-5}{14}$.

$\square$